Non-Abelian Geometric Phases Carried by the Spin Fluctuation Tensor

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When the vector of spin expectation values of a quantum state is transported along a loop in the Bloch ball, we show that the tensor of spin fluctuations picks up a geometric phase. This geometric phase cannot be formulated as a holonomy of loops in the Bloch ball using the standard theory because the space of quantum spin states does not admit a fiber bundle structure over the Bloch ball. Considering spin-1 systems, we formulate this geometric phase as an $SO(3)$ operator by employing small modifications to the standard theory of geometric phase. This $SO(3)$ operator can be extracted from the spin fluctuation tensor. Furthermore, we introduce the notion of *generalized solid angle*, defined for all loops in the Bloch ball in terms of which we interpret this $SO(3)$ geometric phase. Loops that do not pass through the center of the Bloch ball subtend a well-defined solid angle at the center, and we refer to such loops as *non-singular* loops. The loops passing through the center of the Bloch ball are *singular*, in the sense that their solid angle is not well-defined. The generalized solid angle introduced in this paper is well-defined for both singular and non-singular loops, and for the latter, it reduces to the standard solid angle.

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I. INTRODUCTION

The deep relationship between quantum mechanics and geometry is well known. One of the signatures of this relationship, Berry's geometric phase [1], has attracted a renewed interest over the past few years due to its applicability in two different areas — as a topological order parameter characterizing phase transitions [2] and as a phase gate in fault tolerant quantum computation [3]. Although Berry's phase was defined for adiabatic transport of a quantum system along a loop in the Hilbert space, it was later realized that it is a kinematic property that does not depend on the dynamics of the system [4], [5]. That is, geometric phase depends only on the path along which the quantum state was transported and not on the dynamical equation governing the transport or the rate of transport. It has been formulated using a kinematic approach [6], [7]. We adopt such a kinematic approach in this paper. Geometric phase is best defined as a holonomy element of a connection form on a fiber bundle structure imposed on the space of quantum states [8], [9]. For instance, consider Berry's phase defined for loops in the parameter space of a Hamiltonian. The space of ground state eigenvectors of the Hamiltonian has a line bundle structure over this parameter space, assuming that the Hamiltonian is non-degenerate. Berry's phase can be viewed as the holonomy of Berry's connection form on this line bundle [10]. When the Hamiltonian is degenerate, the Berry phase generalizes to a non-Abelian Wilczek-Zee phase, which can also be formulated as a holonomy [11].

Geometric phase is essentially the geometric information stored in the overall phase of the wave-function of a quantum mechanical system. In this paper, we show that geometric information is also stored in second and higher order spin moments of a quantum spin system and we formulate it as a non-abelian geometric phase. We restrict our analysis to pure quantum states i.e., quantum states that can be represented by a vector in the Hilbert space. Corresponding to every pure state, one can define a spin vector in real space as $\vec{s} = (\langle S_x\rangle, \langle S_y\rangle, \langle S_z\rangle)^T$, where S_i are the spin operators and $\langle S_i\rangle$ are the expectation values of the spin operators corresponding to the given pure state. The fluctuations of this spin vector is a rank-2 tensor, given by covariance matrix:

$$\mathbf{T} = \begin{pmatrix} \langle S_x^2\rangle - \langle S_x\rangle^2 & \frac{1}{2}\langle\{S_x,S_y\}\rangle - \langle S_x\rangle\langle S_y\rangle & \frac{1}{2}\langle\{S_x,S_z\}\rangle - \langle S_x\rangle\langle S_z\rangle \\ \frac{1}{2}\langle\{S_x,S_y\}\rangle - \langle S_x\rangle\langle S_y\rangle & \langle S_y^2\rangle - \langle S_y\rangle^2 & \frac{1}{2}\langle\{S_z,S_y\}\rangle - \langle S_z\rangle\langle S_y\rangle \\ \frac{1}{2}\langle\{S_x,S_z\}\rangle - \langle S_x\rangle\langle S_z\rangle & \frac{1}{2}\langle\{S_z,S_y\}\rangle - \langle S_z\rangle\langle S_y\rangle & \langle S_z^2\rangle - \langle S_z\rangle^2 \end{pmatrix} \tag{1}$$

Here, $\{S_i, S_j\} = S_iS_j + S_jS_i$ is the anticommutator of S_i and S_j. Hereafter, we refer to this covariance matrix as the *spin fluctuation tensor*. For a spin-$\frac{1}{2}$ pure state, the spin vector lies in the Bloch sphere, i.e., it has a unit length. For a spin-1 pure state, the spin vector lies in the Bloch ball, i.e., its length can be anywhere in $[0, 1]$. The Bloch ball ($\mathbb{B}$) is a unit ball in $\mathbb{R}^3$.

$$\mathbb{B} = \{\vec{s}:\ \vec{s} \in \mathbb{R}^3,\ |\vec{s}| \leq 1\} \tag{2}$$

When the spin vector is transported along a loop in $\mathbb{B}$, the spin fluctuation tensor picks up a geometric phase. To see this, let us consider the geometric interpretation of the spin fluctuation tensor. It is a symmetric, positive semi-definite matrix and therefore, it has three non-negative eigenvalues and orthogonal eigenvectors. It represents an ellipsoid with the lengths of principle axes given by the eigenvalues and orientation given by the eigenvectors. The pair $(\vec{s}, \mathbf{T})$ can be represented by a vector in $\mathbb{B}$ and an ellipsoid at its tip. The vector represents the spin and the fluctuations of the spin are represented by the ellipsoid (Figure 1(a)). Let us now consider a loop inside $\mathbb{B}$ along which the spin vector is transported. Analogous to the parallel transport of tangent vectors on a sphere, one can consider an intuitive notion of parallel transport of the ellipsoids along the loop, where each of its axes is parallel transported. The ellipsoid will return in a different orientation, capturing the geometric phase of the loop (Figure 1(b)). This geometric phase requires a rigorous definition. In this paper, we formulate this geometric phase as an $SO(3)$ operator and provide a geometric interpretation.

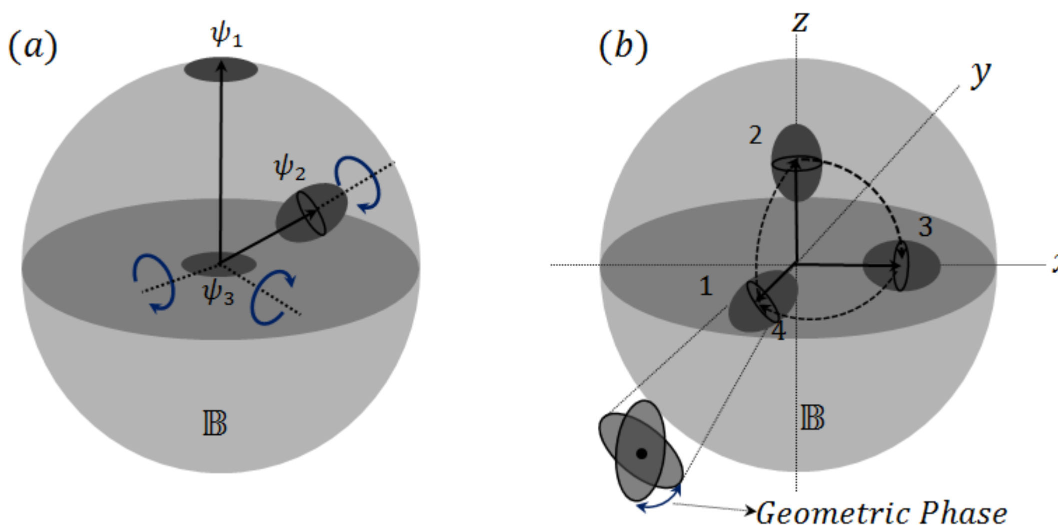

FIG. 1. **Spin vector and fluctuation tensor.** (a) Shows the Bloch ball $\mathbb{B}$ and three quantum states, ψ_1, ψ_2 & ψ_3, each one represented by its spin vector and an ellipsoid representing its spin fluctuation tensor (the ellipsoids are not to scale). ψ_1 has a spin vector of unit length and its spin fluctuation tensor is represented by a disk. ψ_2 has a spin vector of length between 0 & 1 and its spin fluctuation tensor is represented by an ellipsoid. For a fixed spin vector, this ellipsoid has one degree of freedom. ψ_3 has a zero spin vector and its spin fluctuation tensor is represented by a disk at the center. With the spin vector fixed to zero, this disk has two degrees of freedom. (b) Shows an example of a parallel transport of the ellipsoid along a loop and the resulting geometric phase.

The key step in formulating this geometric phase is to rigorously define the parallel transport of the ellipsoids. However, in the following, we show this parallel transport cannot be defined using the standard theory of connections on a fiber bundle. Hereafter, we restrict ourselves to spin-1 systems. The quantum state of a spin-1 system is represented by a vector $\psi = (z_{-1}, z_0, z_{+1})^T$ in a 3-dimensional complex Hilbert space. After normalization and removal of overall phase, the space of all such quantum states is a four dimensional manifold. Topologically, this manifold is a *complex projective plane* ($\mathbb{CP}^2$), defined as the space of all lines in a 3-dimensional complex vector space:

$$\mathbb{CP}^2 = \left\{ (z_{-1}, z_0, z_{+1})^T : (z_{-1}, z_0, z_{+1}) \sim \lambda(z_{-1}, z_0, z_{+1}), \lambda \neq 0 \ \& \ \lambda \in \mathbb{C} \right\} \tag{3}$$

Together, the spin vector and the spin fluctuation tensor contain all the information about a spin-1 quantum state. Indeed, every spin-1 state is uniquely represented by the pair $(\vec{s}, \mathbf{T})$. We may define a map $\phi : \mathbb{CP}^2 \to \mathbb{B}$ that takes every quantum state to its spin vector: $\phi(\psi) = \vec{s}$. In terms of the coordinates, this map is:

$$\phi\left(\begin{pmatrix} z_{-1} \\ z_0 \\ z_{+1} \end{pmatrix}\right) = \begin{pmatrix} \sqrt{2}\mathrm{Re}(z_{-1}z_0^* + z_0 z_{+1}^*) \\ \sqrt{2}\mathrm{Im}(z_{-1}z_0^* + z_0 z_{+1}^*) \\ |z_{+1}|^2 - |z_{-1}|^2 \end{pmatrix} = \vec{s} \in \mathbb{B} \tag{4}$$

However, $\phi : \mathbb{CP}^2 \to \mathbb{B}$ is not a fiber bundle. Any fiber bundle over $\mathbb{B}$ is a product bundle, because $\mathbb{B}$ is a contractible space. $\mathbb{CP}^2$, being 4-dimensional, is not a product bundle over $\mathbb{B}$ because it has a non trivial second homology and any 4-dimensional product bundle over $\mathbb{B}$, being homotopic to the 1-dimensional fiber itself, would have a trivial second homology. Therefore, this geometric phase cannot be formulated as a holonomy of loops in $\mathbb{B}$, in general. Circumventing this difficulty is the first of the two problems that we address in this paper.

Further, the interpretation of this geometric phase poses a separate problem. Berry's phase of a loop in the Bloch sphere is generally interpreted as the solid angle enclosed by the loop [1]. The Bloch sphere is the boundary of the Bloch ball. The definition of solid angles easily extends to loops in the Bloch ball, if they do not pass through the center. A convenient way of imagining this solid angle is to project the loop to the boundary of $\mathbb{B}$, by moving each point on the loop radially outward until it hits the boundary. The projected loop lies on the surface of the ball and it maintains the same solid angle as the original loop (Figure 2 (a)). We refer to such loops as *non-singular* loops. Loops in $\mathbb{B}$ that pass through the center break into discontinuous pieces when projected to the boundary of $\mathbb{B}$; their solid angle cannot be defined using this method and therefore we refer to them as *singular* loops (Figure 2 (b)). It can be intuitively seen that for non-singular loops, this geometric phase is a rotation of the ellipsoid by an angle equal to the solid angle subtended by the loop, about the vector $\vec{s}$, at the base point of the loop. However, interpretation of the geometric phase of singular loops is non-trivial and it is the second problem addressed in this paper. We summarize these two problems as:

(i) Given that $\mathbb{CP}^2$ is not a fiber bundle over $\mathbb{B}$, can we still define a horizontal lift in $\mathbb{CP}^2$, of loops in $\mathbb{B}$ and formulate a definition of geometric phase?

(ii) What is the interpretation of this geometric phase? In particular, can we attach a meaning to "solid angles" for singular loops ?

In Section II we present an outline of the results, and in Section III, we provide details of definitions and proofs of theorems stated in Section II.

II. OUTLINE OF RESULTS

We state our solutions to (i) and (ii) in Section II A and Section II B respectively.

A. Definition of Horizontal Lifts And Geometric Phase

In definitions 1 & 2 below, we answer (i) by invoking the important role played by metrics in the theory of geometric phase [12], [13], [14]. In the definition of the standard Berry's phase and Uhlmann's phase, horizontal lifts are constructed using Berry's connection form [10] and Uhlmann connection form [4], respectively. It has been noted that in both of these cases, the horizontal lift can also be defined as the lift with minimal length in the respective fiber bundles [12]. In general, if the horizontal subspace is defined as the orthogonal complement of the vertical subspace under a Riemannian metric on the fiber bundle, the resulting horizontal lift of a loop always minimizes the length among all lifts of the loop. While $\mathbb{CP}^2$ is not a fiber bundle over $\mathbb{B}$, it has a standard, natural (i.e., maximally symmetric) metric — the Fubini-Study metric. It is essentially the "angle" between two quantum state vectors in the Hilbert space. We define a horizontal lift for loops in $\mathbb{B}$ using this metric.

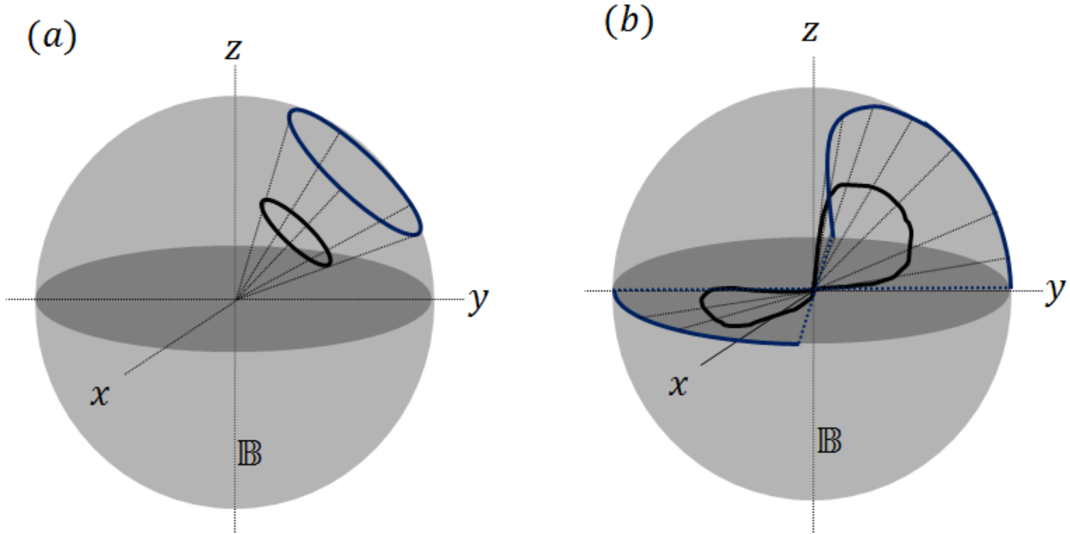

FIG. 2. **Non-singular and singular loops** (a) shows a non-singular loop inside the bloch ball (in black). Its solid angle is equal to the solid angle of its radial projection to the surface (in blue). (b) shows a singular loop inside the bloch ball (in black). This loop cannot be projected to the surface — the center does not have a well-defined image under the projection. The curve in blue is the projection of this loop excluding the point at the center. It does not have a well-defined solid angle.

Definition 1 (Horizontal Lift): A path $\tilde{\gamma} : [0,1] \to \mathbb{CP}^2$ is called a *horizontal lift* of a loop $\gamma : [0,1] \to \mathbb{B}$ iff $\phi \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}$ minimizes the Fubini-Study length in $\mathbb{CP}^2$.

In Section III, we show that the above definition is equivalent to the earlier described intuitive notion of parallel transport of the ellipsoids. We also provide explicit equations to compute the horizontal lift of a given loop in Eq. 29. Before proceeding to define a geometric phase using this horizontal lift, we note that not every loop in $\mathbb{B}$ has a well-defined horizontal lift in $\mathbb{CP}^2$. The relevant regularity conditions on the loop are summarized in theorem 1.

Theorem 1: A continuous, piece-wise differentiable loop $\gamma : [0,1] \to \mathbb{B}$ has a horizontal lift if it is differentiable at every $t \in [0,1]$ where $\gamma(t) = \vec{0} \in \mathbb{B}$.

This theorem essentially states that a loop in $\mathbb{B}$ has a horizontal lift if it has no "kinks" while passing through the center of $\mathbb{B}$. We refer to the loops satisfying the conditions mentioned in this theorem as *liftable* loops. Clearly, any piece-wise differentiable loop not passing through the center of $\mathbb{B}$ is liftable. Figure 3 shows two examples of liftable loops and one example of a loop that is not liftable. Figure 3 (b) is an important example of a loop that appears to have a kink at the center of $\mathbb{B}$, but is nevertheless liftable. The apparent non-differentiability at the center is removable. If we choose the center as the starting and the ending points of the loop, i.e., $\gamma(0) = \gamma(1) = \vec{0} \in \mathbb{B}$, the loop satisfies all conditions mentioned in the theorem. However, the loop in Figure 3 (c) is not liftable. There are multiple points of non-differentiability at the center, and so this loop does not satisfy the conditions mentioned in the above theorem.

We now define geometric phase using the horizontal lift defined above. For a given loop γ, the end points $\tilde{\gamma}(0)$ and $\tilde{\gamma}(1)$ of its horizontal lift are in $\mathbb{CP}^2$ and therefore, there is an operator $U \in SU(3)$ such that $\tilde{\gamma}(1) = U\tilde{\gamma}(0)$. This operator is not unique — there are infinitely many such operators. Through its irreducible representation in $SU(3)$, $SO(3)$ can be regarded as a subgroup of $SU(3)$.

In Section III, we show that there is an $SO(3)$ choice for the operator U, i.e., there is an operator $R \in SO(3) \subset SU(3)$ such that $\tilde{\gamma}(1) = R\tilde{\gamma}(0)$, for every horizontal lift $\tilde{\gamma}$ of a loop γ. However, this operator is still not unique — it has a two fold ambiguity. We clear this ambiguity and provide a more rigorous definition in Section III. We also show that this operator is independent of the choice of $\tilde{\gamma}(0)$, i.e., it is well-defined for γ. We define this $SO(3)$ operator as the geometric phase of γ.

Definition 2 (Geometric Phase): If γ is a liftable loop in $\mathbb{B}$, its *geometric phase* is the operator $R \in SO(3) \subset SU(3)$ such that $\tilde{\gamma}(1) = R\tilde{\gamma}(0)$ for every horizontal lift, $\tilde{\gamma}$ of γ.

In Section III, Eq. 30, we provide an explicit way of computing the geometric phase of a given loop. Going back to the earlier described geometric picture of representing a quantum state by a spin vector and an ellipsoid at its tip, the end points, $\tilde{\gamma}(0)$ and $\tilde{\gamma}(1)$ are two quantum states with the same spin vectors, but different ellipsoids, i.e., we can represent them as $\tilde{\gamma}(0) = (\vec{s}, \mathbf{T_1})$ and $\tilde{\gamma}(1) = (\vec{s}, \mathbf{T_2})$. The geometric phase of γ, we show, is precisely the rotation R which rotates the ellipsoid $\mathbf{T_1}$ to $\mathbf{T_2}$, i.e., $\mathbf{T_2} = R\mathbf{T_1}R^T$.

In the axis-angle representation, a rotation about $\hat{n} \in \mathbb{R}^3$ by an angle $\theta \in [0, 2\pi)$ is represented as $R_{\hat{n}}(\theta)$. For non-singular loops, the geometric phase is $R = R_{\gamma(0)}(\Omega)$, i.e., a rotation about the spin vector $\gamma(0)$ by an angle, Ω, equal to the solid angle of γ (Figure 1 (b)). To see this, we need the following simple facts about the ellipsoids, which follow from Eq. 4. One of the eigenvectors of $\mathbf{T}$ coincides with $\vec{s}$, with an eigenvalue $1 - |\vec{s}|^2$. Therefore, the ellipsoid is always oriented with one axis parallel to $\vec{s}$. The other two eigenvalues are $\frac{1}{2}(1 \pm \sqrt{1 - |\vec{s}|^2})$ and that leaves only one degree of freedom for the ellipsoid when the spin vector is fixed, i.e., rotation about the spin vector (Figure 1 (a)). Therefore, if $\gamma(t) \neq 0$ throughout the loop, the geometric phase is necessarily a rotation about the vector $\gamma(0)$. The parallel transport of the ellipsoid is reminiscent of the parallel transport of a tangent line to S^2 along a loop and thus, the holonomy is the solid angle of the loop. Therefore, the angle of rotation of the ellipsoid is also this solid angle.

However, the above interpretation does not work for singular loops; the geometric phase is more non trivial. We provide a generalization of the above interpretation in the following section.

B. Interpretation of Geometric Phase

To answer (ii), we define *generalized solid angles* for all loops inside $\mathbb{B}$ in definition 3 below using which we interpret the geometric phase. This generalized solid angle of a loop in the Bloch ball is defined by first projecting the loop to the *real projective plane* ($\mathbb{RP}^2$) and then defining a solid angle for the projection in $\mathbb{RP}^2$. We begin with the definition of the projection.

$\mathbb{RP}^2$ is the space of all lines through the center of $\mathbb{R}^3$. Equivalently, it is the space obtained by identifying diametrically opposite points on a sphere (S^2). We use the following notation for points in $\mathbb{RP}^2$:

Notation: For a unit vector $\hat{n} \in S^2$, its projection in $\mathbb{RP}^2$, i.e., the equivalence class $\{+\hat{n}, -\hat{n}\}$ will be denoted by $\pm\hat{n}$.

Every loop in S^2 can be projected to a loop in $\mathbb{RP}^2$. As described earlier, the solid angle of a non-singular loop can be pictured by projecting it to the boundary of $\mathbb{B}$, which is an S^2 (Figure 2 (a)). A singular loop can also be projected to S^2 after removing the point(s) at the center. However, the projected path will be discontinuous (Figure 2 (b)). The points of discontinuties are always diametrically opposite. Every time the loop crosses the center of $\mathbb{B}$, the projected path makes discontinuous jump across the diameter of $\mathbb{B}$, parallel to the tangent of the loop at the center. This holds for all liftable loops. This discontinuity can be removed by identifying diametrically opposite points on S^2 and doing so, we obtain an $\mathbb{RP}^2$. Thus, every liftable loop γ in $\mathbb{B}$ can be projected to

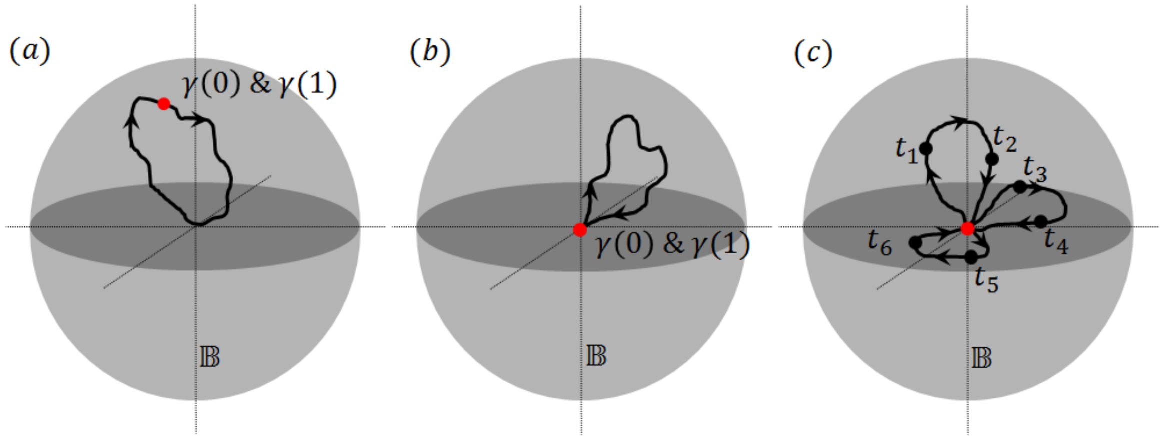

FIG. 3. **liftable and unliftable loops.** (a) and (b) show liftable loops and (c) shows an unliftable loop. In all three loops, the red point represents the starting and the ending point (i.e., $\gamma(0)$ and $\gamma(1)$). For the loop in (a), $\gamma^{-1}(\vec{0}) = \{t\}$ for some $t \in (0,1)$ and the loop is differentiable at that point. The loop in (b) has a kink at zero, but with a suitable choice of the starting an ending points, it is liftable. In particular, when the starting and the ending points are chosen at the center, i.e., $\gamma^{-1}(\vec{0}) = \{0,1\}$, $\dot{\gamma}(0)$ and $\dot{\gamma}(1)$ are both well-defined and therefore, the loop is liftable. The loop in (c) has multiple kinks at the center. Six intermediate points between 0 & 1, with the following ordering: $0 < t_1 < t_2 < t_3 < t_4 < t_5 < t_6 < 1$ are indicated to guide the reader through the loop. There is no choice of the starting and the ending points such that it is liftable. $\gamma^{-1}(\vec{0})$ has two points other than 0 & 1 and the loop is not differentiable at either of these points. Therefore, this loop is not liftable.

a continuous path $\alpha : [0,1] \to \mathbb{RP}^2$:

$$\alpha(t) = \begin{cases} \pm\frac{\gamma(t)}{|\gamma(t)|} & \gamma(t) \neq 0 \\ \pm\frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|} & \gamma(t) = 0 \end{cases} \tag{5}$$

Note that this projection is an open path in general.

We next define a solid angle for paths in $\mathbb{RP}^2$, as an appropriate $U(1)$ holonomy. The lens space $L(4,1)$ is a $U(1)$ bundle over $\mathbb{RP}^2$ and S^2. In fact, this is the only lens space that is a $U(1)$ bundle over $\mathbb{RP}^2$ [15]. It is defined as a quotient of the 3-sphere (S^3). S^3 can be represented as the set of all normalized vectors in $\mathbb{C}^2$, i.e.,

$$S^3 = \{(z_1, z_2) : \quad z_1, z_2 \in \mathbb{C}, \ \ |z_1|^2 + |z_2|^2 = 1\} \tag{6}$$

$L(4,1)$ is obtained by identifying the orbit of $Z_4 = \{1, i, -1, -i\}$ in S^3, i.e.,
$\{(z_1, z_2), (iz_1, iz_2), (-z_1, -z_2), (-iz_1, -iz_2)\}$.

$$L(4,1) = S^3/_{(z_1,z_2)\sim(iz_1,iz_2)} \tag{7}$$

The solid angle of a loop in S^2 can be defined as its $U(1)$ holonomy after lifting it to $L(4,1)$. Similarly, we define the solid angle of a loop in $\mathbb{RP}^2$ also as its $U(1)$ holonomy after lifting it to $L(4,1)$. An important property of this solid angle is that it is preserved under the projection map from S^2 to $\mathbb{RP}^2$ — the solid angle of a loop in S^2 is equal to the solid angle of its projection in $\mathbb{RP}^2$.

We prove this in lemma 3 in Section III B. The appropriate generalization of a holonomy to open paths is a *vertical displacement* [16]. The vertical displacement of a path in $\mathbb{RP}^2$ is a map from the fiber above the initial point of the path to the fiber above the final point of the path. Noting that $SO(3) \approx L(2,1)$ is a double cover of $L(4,1)$ and it acts transitively on $L(4,1)$, the vertical displacement can be represented by an $SO(3)$ action on $L(4,1)$, i.e., an operator $V \in SO(3)$. We provide the details in Section III.

We now define the generalized solid angle of a loop in $\mathbb{B}$.

Definition 3 (Generalized Solid Angle): Let γ be a liftable loop in $\mathbb{B}$ and α be its projection in $\mathbb{RP}^2$. If $\tilde{\alpha}$ is a horizontal lift of α in $L(4,1)$ with a vertical displacement $V \in SO(3)$ and $\hat{k}$ is any unit vector normal to both $\alpha(0)$ and $\alpha(1)$, the generalized solid angle (Ω) of the loop γ is given by $\Omega = \cos^{-1}(\hat{k} \cdot V\hat{k})$.

In Section III C, we show that the expression $\Omega = \cos^{-1}(\hat{k} \cdot V\hat{k})$ is the correct holonomy of α when it is closed, and it is equal to the standard solid angle of γ when it is non-singular. Furthermore, we also show that the same expression is a meaningful definition of the solid angle of α, even when it is open. Hence we refer to this angle as the generalized solid angle of γ. The following theorem establishes the connection between the generalized solid angle and geometric phase:

Theorem 2: If γ is a liftable loop in $\mathbb{B}$ and α is its projection in $\mathbb{RP}^2$, then the geometric phase of γ is equal to the vertical displacement of α.

In the following section, we fill in the details of definitions 1, 2 & 3 and provide a proof of theorem 1 and theorem 2.

III. FORMULATION AND PROOFS OF THEOREM 1 AND THEOREM 2

The basic idea behind the proof of theorem 1 is that although $\phi : \mathbb{CP}^2 \to \mathbb{B}$ does not have a fiber bundle structure, it is closely related to a fiber bundle. In fact, it can be constructed as a quotient of a fiber bundle. $\mathbb{B}$ can be constructed from $S^2 \times [0,1]$ by collapsing the sphere $S^2 \times \{0\}$ to a point. We show in lemma 2(a) below that $\mathbb{CP}^2$ can be constructed from $L(4,1) \times [0,1]$ by collapsing $L(4,1) \times \{0\}$ and $L(4,1) \times \{1\}$ to an $\mathbb{RP}^2$ and an S^2 respectively. $L(4,1) \times [0,1]$ is an S^1 bundle over $S^2 \times [0,1]$, because $L(4,1)$ is an S^1 bundle over S^2. Thus, $\mathbb{CP}^2 \to \mathbb{B}$ can be constructed from the fiber bundle $L(4,1) \times [0,1] \to S^2 \times [0,1]$. Before proceeding to state and prove lemma 2, we develop a geometrical construction of $L(4,1)$; we show in lemma 1 that $L(4,1)$ is the space of all tangent lines to a unit sphere.

Lemma 1: $L(4,1)$ is homeomorphic to the space of all tangent lines to a unit sphere and it is an S^1 bundle over both S^2 and $\mathbb{RP}^2$.

Proof: A tangent line (ℓ) to a sphere is uniquely represented by the pair $\ell = (\hat{v}, \pm\hat{u})$ (Figure 4(a)) of orthogonal unit vectors, $\hat{v}$ representing the point of tangency of ℓ and $\hat{u}$ representing the direction of ℓ. Here, $-\hat{u}$ and $+\hat{u}$ represent the same tangent line and therefore, we use a " $\pm$ " sign before $\hat{u}$. We show that the space of all tangent lines to a sphere, i.e, $\{\ell = (\hat{v}, \pm\hat{u}) : \quad \hat{v} \cdot \hat{u} = 0\}$ is homeomorphic to $L(4,1)$ by explicitly constructing a 4-sheeted covering map from S^3 to this space.

Noting that $SU(2)$ is topologically homeomorphic to S^3 and $SO(3)$ acts transitively on the space of tangent lines to a sphere, we construct a composition of the following two maps:

$$SU(2) \xrightarrow{f} SO(3) \xrightarrow{g} \{\ell = (\hat{v}, \pm\hat{u}) : \quad \hat{v} \cdot \hat{u} = 0\} \tag{8}$$

f is the standard double cover from $SU(2)$ to $SO(3)$ i.e., $f : e^{i\hat{n}\cdot\vec{\sigma}\frac{\theta}{2}} \mapsto R_{\hat{n}}(\theta) \in SO(3)$, where $\hat{n}$ is a unit vector in $\mathbb{R}^3$ and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{9}$$

The map g is constructed out of the action of $SO(3)$ on the tangent lines to a sphere. Fixing a tangent line $\ell_0 = (\hat{z}, \pm\hat{x})$ (Figure 4(a)), we obtain:

$$g : R_{\hat{n}}(\theta) \mapsto R_{\hat{n}}(\theta)\ell_0 = (R_{\hat{n}}(\theta)\hat{z}, \pm R_{\hat{n}}(\theta)\hat{x}) \tag{10}$$

We now show that $g \circ f : SU(2) \to \{(\hat{v}, \pm\hat{u}) : \quad \hat{v} \cdot \hat{u} = 0\}$ is the required 4-sheet covering map. The action of $SO(3)$ on a tangent line has a Z_2 stabilizer. For instance, the stabilizer of ℓ_0 is $\{1, R_{\hat{z}}(\pi)\}$. Therefore, g is a double covering map. For an arbitrary tangent line ℓ, the pre-image set under g contains two points in $SO(3)$. If $\ell = R_{\hat{n}}(\theta)\ell_0$, for some $\hat{n}$ and θ, then its pre-image set is $g^{-1}(\ell) = \{R_{\hat{n}}(\theta), R_{\hat{n}}(\theta)R_{\hat{z}}(\pi)\}$. Further, $f^{-1} \circ g^{-1}(\ell)$ is a set of 4 elements in $SU(2)$ given by:

$$f^{-1} \circ g^{-1}(\ell) = e^{i\hat{n}\cdot\vec{\sigma}\frac{\theta}{2}}\{1, i\sigma_z, -1, -i\sigma_z\} \tag{11}$$

Thus, $g \circ f$ is the required covering map and therefore, $L(4,1) \approx \{\ell = (\hat{v}, \pm\hat{u}) : \quad \hat{v} \cdot \hat{u} = 0\}$. We can now define the bundle maps $\pi_1 : L(4,1) \to S^2$ and $\pi_2 : L(4,1) \to \mathbb{RP}^2$:

$$\begin{aligned} \pi_1(\ell = (\hat{v}, \pm\hat{u})) &= \hat{v} \in S^2 \\ \pi_2(\ell = (\hat{v}, \pm\hat{u})) &= \pm\hat{u} \in \mathbb{RP}^2 \end{aligned} \tag{12}$$

π_1 takes every tangent line to its point of tangency, and π_2 takes every tangent line to a parallel line through the center, which is an element of $\mathbb{RP}^2$. It is straight forward to verify that they are both S^1 bundle maps ■.

A natural metric on $L(4,1)$ is induced by the round metric on S^3. This metric, at a point $\ell = (\hat{v}, \pm\hat{u}) \in L(4,1)$ is:

$$ds^2 = d\hat{v} \cdot d\hat{v} + d\hat{u} \cdot d\hat{u} - (\hat{v} \cdot d\hat{u})^2 \tag{13}$$

The first term $(d\hat{v} \cdot d\hat{v})$ corresponds to the distance covered by the point of contact on S^2. The term $d\hat{u} \cdot d\hat{u} - (\hat{v} \cdot d\hat{u})^2$ corresponds to the angle of rotation of the tangent line about its point of contact.

Using a similar argument, it can be shown that the lens space $L(2,1)$ is the space of all unit tangent vectors to a unit sphere, i.e., $L(2,1) \approx \{(\hat{v}, \hat{u}) : \quad \hat{u} \cdot \hat{v} = 0\}$ (Figure 4 (b)).

Lemma 2:

(a) $\mathbb{CP}^2$ can be constructed from the stack $L(4,1) \times [0,1]$ by collapsing $L(4,1) \times \{0\}$ to an $\mathbb{RP}^2$ and $L(4,1) \times \{1\}$ to an S^2 using the respective bundle maps π_1 and π_2. That is,

$$\mathbb{CP}^2 = {}^{L(4,1)\times[0,1]}/_{\pi} \tag{14}$$

where $\pi = 1$ on $L(4,1) \times (0,1)$, $\pi = \pi_1$ on $L(4,1) \times \{1\}$ and $\pi = \pi_2$ on $L(4,1) \times \{0\}$

(b) Writing $\mathbb{B}^\circ - \{0\} = S^2 \times (0,1)$, where $\mathbb{B}^\circ$ is the interior of $\mathbb{B}$, the restriction of ϕ to $L(4,1) \times (0,1)$ is :

$$\phi = \pi_1 \times 1 : L(4,1) \times (0,1) \to S^2 \times (0,1). \tag{15}$$

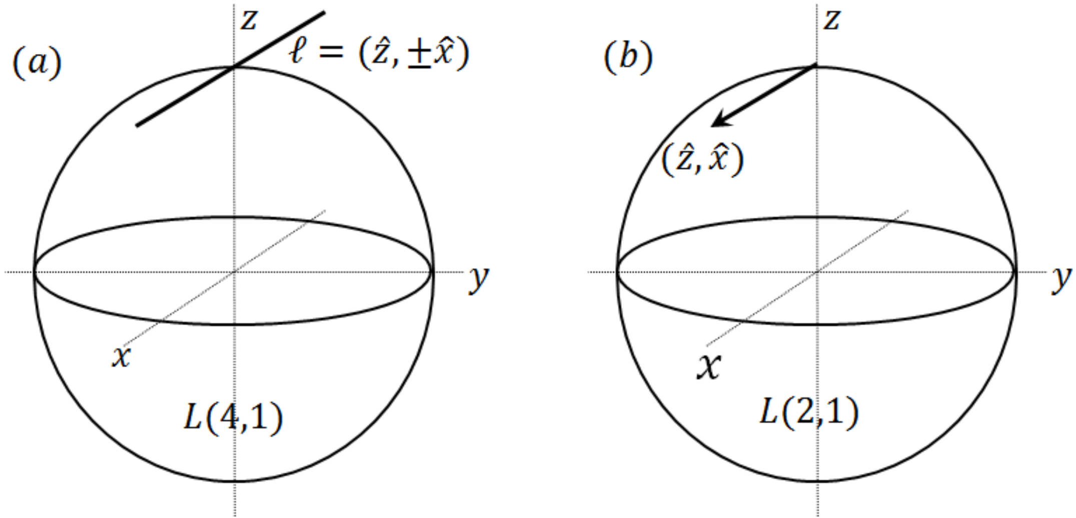

FIG. 4. **The lens spaces** $L(4,1)$ **and** $L(2,1)$**.** $L(4,1)$ is the space of all tangent lines to a sphere and $L(2,1)$ is the space of all unit tangent vectors to a sphere. (a) shows the tangent line $\ell = (\hat{z}, \pm\hat{x}) \in L(4,1)$, parallel to $\hat{x}$ and touching the sphere at $\hat{z}$. (b) shows a unit tangent vector to a sphere $(\hat{z}, \hat{x}) \in L(2,1)$.

(c) $\mathbb{CP}^2$ is the space of all chords to a unit sphere and ϕ maps each chord to its center.

Proof: We begin with a proof of (a). Let us consider the pre-image sets of ϕ:

$$\phi^{-1}(\vec{s}) = \begin{cases} \mathbb{RP}^0 \text{ if } |\vec{s}| = 1 \\ \mathbb{RP}^1 \text{ if } 0 < |\vec{s}| < 1 \\ \mathbb{RP}^2 \text{ if } |\vec{s}| = 0 \end{cases} \tag{16}$$

This can be shown using the explicit algebraic expression of ϕ, Eq. 4. However, it is more illuminating to use the earlier described geometric picture of representing a point in $\mathbb{CP}^2$ as a vector and an ellipsoid, i.e., $(\vec{s}, \mathbf{T})$ (Figure 1 (a)). The lengths of the axes of the ellipsoid are $1 - |\vec{s}|^2$, $\frac{1}{2}(1 \pm \sqrt{1 - |\vec{s}|^2})$. Therefore, its dimensions depend only on the length of the spin vector. Furthermore, one of its axes is parallel to $\vec{s}$, when $|\vec{s}| \neq 0$. For a given spin vector with $0 < |\vec{s}| < 1$, the ellipsoid has one degree of freedom — rotation about $\vec{s}$, which produces the set of all quantum states with spin vector $\vec{s}$. This set is an $\mathbb{RP}^1$, because the ellipsoid has a two fold symmetry when rotated about $\vec{s}$.

On the boundary of $\mathbb{B}$, when $|\vec{s}| = 1$, the two transverse axes of the ellipsoid are equal, i.e., the ellipsoid degenerates into a disk perpendicular to $\vec{s}$. It has no degrees of freedom; it is the only quantum state with the given spin vector. Thus, the pre-image set of this spin vector is just a point i.e., $\mathbb{RP}^0$.

Finally, when $|\vec{s}| = 0$, the ellipsoid again degenerates to a disk at the center of $\mathbb{B}$. This time, however, it has two degrees of freedom. The pre-image set $\phi^{-1}(0)$ is the space of all orientations of a disk in $\mathbb{R}^3$ centered at the origin. This is indeed $\mathbb{RP}^2$.

It follows, now, that the pre-image set of the boundary of $\mathbb{B}$, i.e., $\phi^{-1}(\{\vec{s} : \quad |\vec{s}| = 1\})$ is a sphere in $\mathbb{CP}^2$. For a shell of radius $0 < r < 1$, the pre-image set is a lens space $L(4,1)$:

$$\phi^{-1}(\{\vec{s}: \quad |\vec{s}| = r\}) = L(4,1) \quad 0 < r < 1 \tag{17}$$

To show this, we use lemma 1 and construct a bijective map from the pre-image of the shell to $L(4,1)$. Consider the map $(\vec{s}, \mathbf{T}) \mapsto (\hat{v}, \pm\hat{u})$ where $\hat{v} = \frac{\vec{s}}{r}$ and $\hat{u}$ is the eigenvector of $\mathbf{T}$ normal to $\vec{s}$, with the larger eigenvalue. Indeed, there is a one-one correspondence between the orientations of an ellipsoid at $\vec{s}$ and tangent lines at $\vec{s}$ to a sphere of radius $|\vec{s}|$. Thus, the pre-image of a shell is homeomorphic to $L(4,1)$.

We can now construct $\mathbb{CP}^2$ using the pre-image sets:

$$\begin{aligned} \phi^{-1}(\{\vec{s}: \quad |\vec{s}| = 1\}) &= S^2 \\ \phi^{-1}(\{\vec{s}: \quad 0 < |\vec{s}| < 1\}) &= L(4,1) \times (0,1) \\ \phi^{-1}(\vec{0}) &= \mathbb{RP}^2 \end{aligned} \tag{18}$$

$\mathbb{CP}^2$ is therefore obtained by attaching an $\mathbb{RP}^2$ and an S^2 to either ends of $L(4,1) \times (0,1)$. The attaching maps are easily seen to be π_1 and π_2, using the geometric picture. Thus, $\mathbb{CP}^2$ is obtained from $L(4,1) \times [0,1]$ by collapsing $L(4,1) \times \{0\}$ to an $\mathbb{RP}^2$ and $L(4,1) \times \{1\}$ to an S^2 using the respective bundle maps.

(b) follows trivially from the above construction of pre-image sets. The geometrical construction claimed in (c) can be shown as follows. The chords passing through the center of a unit sphere form an $\mathbb{RP}^2$. The chords at some distance $r \in (0,1)$ from the center form an $L(4,1)$ and the chords at a distance 1 from the center degenerate to points on a sphere, forming a sphere. Thus, the space of all chords to a unit sphere has the same structure as $\mathbb{CP}^2$ and is homemorphic to it. ■.

Lemma 2(c) is also a consequence of Majorana constellation [17] which has been used very fruitfully to understand geometric phases [18]. States of a spin 1 system can be considered as symmetric states of a two coupled spin half systems. A spin half state is a point on a Bloch sphere (i.e., $\mathbb{CP}^1$) and therefore, a spin 1 state is a symmetrized pair of points on the Bloch sphere. This is equivalent to a chord[19]. ϕ maps each chord to its center.

We can represent a chord as $(r, \hat{v}, \pm\hat{u})$, where $r\hat{v}$ is the center of the chord and $\hat{u}$ is its direction. This corresponds to a quantum state whose spin vector is $r\hat{v}$ and the ellipsoid is oriented with the major axis parallel to $\hat{u}$. It is straightforward to construct this quantum state $\psi \in \mathbb{C}^3$. For instance, written in the standard z-basis,

$$\psi = \left(\sqrt{\frac{1-r}{2}}, 0, \sqrt{\frac{1+r}{2}}\right) \mapsto (r, \hat{z}, \pm\hat{x}) \tag{19}$$

Quantum states corresponding to any chord can be obtained by preforming rotations on both sides of the above equation. Conversely, the chord corresponding to given quantum state can be obtained from its spin vector and fluctuation tensor, $(\vec{s}, \mathbf{T})$ — it is the chord centered at $\vec{s}$ and oriented parallel to the largest axis of $\mathbf{T}$ perpendicular to $\vec{s}$.

The Fubini-Study metric on $\mathbb{CP}^2$ can be applied to the space of all chords to a unit sphere. At $(r, \hat{v}, \pm\hat{u})$, the metric is:

$$ds^2 = \frac{1}{2}(1 - \sqrt{1-r^2}) d\hat{v} \cdot d\hat{v} + \sqrt{1-r^2}(\hat{u} \cdot d\hat{v})^2 + (1-r^2)(d\hat{u} \cdot d\hat{u} - (\hat{v} \cdot d\hat{u})^2) + \frac{1}{4(1-r^2)} dr^2 \tag{20}$$

We now proceed to prove theorem 1.

A. Proof of theorem 1

Without loss of generality, we may assume that $\dot{\gamma}(t) \neq 0$ whenever it is well-defined. Therefore, $\gamma^{-1}(\vec{0})$ is a zero dimensional compact manifold, i.e., it is a finite set of points. Adding the end points 0 & 1 to this finite set, we obtain a set of points, $\gamma^{-1}(0) \cup \{0,1\} = \{a_0, \cdots a_{n+1}\}$ where, $a_i < a_{i+1}$ and $a_0 = 0$ & $a_{n+1} = 1$. This set divides the loop into $n+1$ pieces, $\gamma_j : [a_{j-1}, a_j] \to \mathbb{B}$ for $j = 1, 2, \cdots n+1$. Each piece γ_j may start and end at the center of $\mathbb{B}$, but lies away from the center otherwise. That is, its interior lies away from the center, $\gamma((a_{j-1}, a_j)) \subset S^2 \times (0,1]$. The closure of this path in $S^2 \times [0,1]$ has a horizontal lift in $L(4,1) \times [0,1]$, defined using the standard theory of connections [16], because this space has a circle bundle structure over $S^2 \times [0,1]$. We denote this horizontal lift by $\tilde{\gamma}_j : [a_{j-1}, a_j] \to L(4,1) \times [0,1]$. This path can be projected to $\mathbb{CP}^2$ by composing it with π, as shown in lemma 2(a). The idea behind this proof is to show that these projected paths can be attached continuously under the assumptions of the theorem, and the resulting path in $\mathbb{CP}^2$ is a lift of γ that minimizes the Fubini-Study length.

Within (a_{j-1}, a_j), we may write $\gamma_j(t) = (\frac{\gamma_j(t)}{|\gamma_j(t)|}, |\gamma_j(t)|) \in S^2 \times (0,1]$ where the two components represent the coordinates in S^2 and $(0,1]$ respectively, i.e., $\frac{\gamma_j(t)}{|\gamma_j(t)|} \in S^2$ and $|\gamma_j(t)| \in (0,1]$. Let us define the closure of the first component as $\beta_j : [a_{j-1}, a_j] \to S^2$:

$$\beta_j(t) = \begin{cases} \frac{\gamma_j(t)}{|\gamma_j(t)|} & a_{j-1} < t < a_j \\ \lim_{t' \to a_k} \frac{\gamma_j(t')}{|\gamma_j(t')|} & t = a_k, \; k = j, j-1 \end{cases} \tag{21}$$

Let $\tilde{\beta}_j$ denote a horizontal lift of β_j in $L(4,1)$. We define paths $\tilde{\gamma}_j : [a_{j-1}, a_j] \to L(4,1) \times [0,1]$ as:

$$\tilde{\gamma}_j(t) = (\tilde{\beta}_j(t), |\gamma_j(t)|) \tag{22}$$

We next show that after projecting these paths to $\mathbb{CP}^2$, i.e., $\pi \circ \tilde{\gamma}_j$ can be attached continuously at all a_j for $j = 1, 2 \cdots n$. Note that $\gamma(a_j) = \vec{0}$ for $j = 1, 2 \cdots n$. The end points of the two neighboring paths, $\tilde{\gamma}_j$ and $\tilde{\gamma}_{j+1}$ at a_j, projected to $\mathbb{CP}^2$ are given by:

$$\begin{aligned} \pi \circ \tilde{\gamma}_j(a_j) = \pi \circ (\tilde{\beta}_j(a_j), 0) \equiv \pi_2 \circ \tilde{\beta}_j(a_j) \in \mathbb{RP}^2 = \phi^{-1}(\vec{0}) \\ \pi \circ \tilde{\gamma}_{j+1}(a_j) = \pi \circ (\tilde{\beta}_{j+1}(a_j), 0) \equiv \pi_2 \circ \tilde{\beta}_{j+1}(a_j) \in \mathbb{RP}^2 = \phi^{-1}(\vec{0}) \end{aligned} \tag{23}$$

It suffices to show that the first point of the lift, $\tilde{\beta}_{j+1}(a_j)$, can be chosen such that the above two points coincide in $\mathbb{CP}^2$. We begin with a simple observation; since γ is liftable, it is differentiable at a_j and it follows that [20]:

$$\begin{aligned} \beta_j(a_j) = \lim_{t \to a_j} \frac{\gamma_j(t)}{|\gamma_j(t)|} = \frac{\dot{\gamma}(a_j)}{|\dot{\gamma}(a_j)|} \\ \beta_{j+1}(a_j) = \lim_{t \to a_j} \frac{\gamma_{j+1}(t)}{|\gamma_{j+1}(t)|} = -\frac{\dot{\gamma}(a_j)}{|\dot{\gamma}(a_j)|} \end{aligned} \tag{24}$$

Let $\tilde{\beta}_j(a_j) = (\beta_j(a_j), \pm\hat{u}) \in L(4,1)$ for some $\hat{u}$ normal to $\beta_j(a_j)$, following lemma 1. We may choose

$$\tilde{\beta}_{j+1}(a_j) = (\beta_{j+1}(a_j), \pm\hat{u}) \in \pi_1^{-1}(\beta_{j+1}(a_j)) \tag{25}$$

This is a valid choice because $\hat{u}$ is normal to $\beta_{j+1}(a_j)$ (this follows from $\beta_{j+1}(a_j) = -\beta_j(a_j)$). It now follows that $\pi_2 \circ \beta_j(a_j) = \pi_2 \circ \beta_{j+1}(a_j) = \pm\hat{u} \in \mathbb{RP}^2$ and therefore, $\tilde{\gamma}_j$ and $\tilde{\gamma}_{j+1}$ can be attached continuously .

It remains to show that the lift $\tilde{\gamma}$ obtained by attaching $\pi \circ \tilde{\gamma}_j$ minimizes the Fubini-Study metric. It suffices to show this for the interior of each segment $\pi \circ \tilde{\gamma}_j$, which is contained in $L(4,1) \times (0,1)$. Consider $\tilde{\gamma}_j(t) = (r(t), \hat{v}(t), \pm\hat{u}(t))$ as a path in the set of all chords to a unit sphere, using the notation $(r, \hat{v}, \pm\hat{u})$ for a chord with center at $r\hat{v}$ and in direction $\hat{u}$. It follows from the construction of $\tilde{\gamma}_j$ that:

$$\begin{aligned} r(t) &= |\gamma_j(t)| \\ (\hat{v}(t), \pm\hat{u}(t)) &= \tilde{\beta}_j(t) \in L(4,1) \quad \text{and} \\ \hat{v}(t) &= \beta_j(t) \end{aligned} \tag{26}$$

$r(t)$ and $\hat{v}(t)$ are determined by $|\gamma_j(t)|$ and $\beta_j(t)$ respectively. The key observation is that the horizontal lift $\tilde{\beta}_j$ minimizes the length under the induced round metric on $L(4,1)$ (Eq. 13) among all lifts of β_j [16], [21]. That is, $\hat{u}(t)$ is chosen so as to minimize the length of $\tilde{\beta}_j$ in $L(4,1)$. From Eq. 13, it follows that $\dot{\hat{u}} \cdot \dot{\hat{u}} = (\hat{v} \cdot \dot{\hat{u}})^2$ — this is the condition for minimizing the length. From Eq. 20, it follows that the same condition minimizes the Fubini-Stuidy length of $\tilde{\gamma}_j$ in $L(4,1) \times (0,1)$. Thus, $\tilde{\gamma}$ is a horizontal lift of γ. ■

The intuitive notion of parallel transport of the ellipsoids discussed in Section I corresponds to a minimization of the induced round metric on $L(4,1)$. Thus, it follows that this intuitive notion is consistent with the definition of the horizontal lift.

Geometric phase was defined in Section II as the operator $R \in SO(3) \subset SU(3)$ such that $\tilde{\gamma}(1) = R\tilde{\gamma}(0)$. However, this operator is not unique — it has a two fold ambiguity because $\tilde{\gamma}(0)$ has a non-trivial stabilizer in $SO(3)$. For instance, when $|\vec{s}| \neq 0$, $R_{\vec{s}}(\pi)\tilde{\gamma}(0) = \tilde{\gamma}(0)$. We now use the details of the construction of $\tilde{\gamma}$ to provide a rigorous definition of R.

Corresponding to each segment β_j in S^2, we define a vertical displacement $R_j \in SO(3)$ such that its lift satisfies $\tilde{\beta}_j(a_j) = R_j\tilde{\beta}_j(a_{j-1})$. Here, $\tilde{\beta}_j$ is considered a path in the space of tangent lines to a sphere and R_j acts on the tangent lines as a rotation. To define R_j uniquely, we note that $SO(3) \approx L(2,1)$ is a double cover of $L(4,1)$. As remarked earlier, $L(2,1)$ is the space of all unit tangent vectors to a unit sphere. $\tilde{\beta}_j$ can be lifted to $L(2,1)$, and the end points of this lift will define a unique $R_j \in SO(3)$. For example, if $\tilde{\beta}_j(t) = (\hat{v}(t), \pm\hat{u}(t))$, we may assume without loss of generality, that $(\hat{v}(t), \hat{u}(t))$ represents a continuous path in the space of all unit tangent vectors, i.e., in $L(2,1)$. Indeed, this is a lift of $\tilde{\beta}_j$ in $L(2,1)$. The only other lift is $(\hat{v}(t), -\hat{u}(t))$. Both of these lifts define the same, unique vertical displacement $R_j \in SO(3)$ with

$$R_j\hat{v}(a_{j-1}) = \hat{v}(a_j) \quad \text{and} \quad R_j\hat{u}(a_{j-1}) = \hat{u}(a_j) \tag{27}$$

Noting that $L(4,1)$ is a $U(1)$ bundle over S^2, it follows that this operator is independent of the choice of the first point, $\tilde{\beta}_j(a_{j-1})$ of the lift [16]. We now define the geometric phase as

$$R = R_{n+1}R_n \cdots R_1 \tag{28}$$

It follows that $R\tilde{\gamma}(0) = \tilde{\gamma}(1)$.

We end this section with an explicit formula to compute the horizontal lift in $\mathbb{CP}^2$ and the geometric phase of a given loop in $\mathbb{B}$. It suffices to compute $\tilde{\beta}_j$ and R_j for each segment γ_j of the loop. Assuming that $\beta_j = \hat{v}(t)$ for $t \in [a_j, a_{j+1}]$, we are to find a $\hat{u}(t)$ such that $(\hat{v}(t), \pm\hat{u}(t))$ is a horizontal lift of β_j. Using the minimization condition for Eq. 13 and $\hat{u}(t) \cdot \hat{v}(t) = 0$, it follows that $\hat{u}(t)$ is the solution to the differential equation:

$$\frac{d}{dt}\hat{u}(t) = -\left(\frac{d\hat{v}(t)}{dt} \cdot \hat{u}(t)\right)\hat{v}(t) \tag{29}$$

To find the geometric phase, we introduce $X : [a_j, a_{j+1}] \to SO(3)$ satisfying $\hat{u}(t) = X(t)\hat{u}(a_j)$, $\hat{v}(t) = X(t)\hat{v}(a_j)$ and $X(a_j) = 1$. The geometric phase will then be $R_j = X(a_{j+1})$. It is straightforward to see that $X(t)$ is the solution to the following initial value problem:

$$\begin{aligned} &\frac{d}{dt}X(t) = \left(\frac{d\hat{v}(t)}{dt}\hat{v}(t)^T - \hat{v}(t)\frac{d\hat{v}(t)}{dt}^T\right)X \\ &X(a_j) = 1 \end{aligned} \tag{30}$$

The above two equations, along with Eq. 19 provide a complete set of equations to compute the horizontal lift and the geometric phase for any loop in $\mathbb{B}$. Next, we prove theorem 2.

B. Proof of Theorem 2

As shown in lemma 1, $L(4,1)$ admits two S^1 bundle structures, namely, $\pi_1 : L(4,1) \to S^2$ and $\pi_2 : L(4,1) \to \mathbb{RP}^2$. Accordingly, loops in S^2 and loops in $\mathbb{RP}^2$ both have well-defined solid angles in terms of the respective $U(1)$ holonomies. The natural projection from S^2 to $\mathbb{RP}^2$ preserves the solid angle. This is the core ingredient in the interpretation of the geometric phase and the proof of theorem 2. We prove this fact in lemma 3 and then proceed to prove theorem 2. We denote the natural projection map from S^2 to $\mathbb{RP}^2$ by p.

Lemma 3: Let β be a piece-wise differentiable path in S^2 and $p \circ \beta$ be its projection in $\mathbb{RP}^2$. The vertical displacements of the horizontal lifts of β and $p \circ \beta$ in $L(4,1)$ are equal.

Proof: Let $\beta(t) = \hat{v}(t)$ and let $\tilde{\beta}(t) = (\hat{v}(t), \pm\hat{u}(t))$ be its horizontal lift in $L(4,1)$. The projection of β in $\mathbb{RP}^2$ is $p \circ \beta = \pm\hat{v}(t)$. We first show that the path obtained by interchanging $\hat{u}$ and $\hat{v}$, i.e., $(\hat{u}(t), \pm\hat{v}(t))$ is a horizontal lift of $p \circ \beta$ in $L(4,1)$.

From the condition $\hat{u}(t) \cdot \hat{v}(t) = 0$, it follows that $\dot{\hat{u}}(t) \cdot \hat{v}(t) + \hat{u}(t) \cdot \dot{\hat{v}}(t) = 0$. Therefore, the paths $(\hat{v}(t), \pm\hat{u}(t))$ and $(\hat{u}(t), \pm\hat{v}(t))$ have the same length in $L(4,1)$(see Eq. 13). Further, $(\hat{u}(t), \pm\hat{v}(t))$ is a lift of $p \circ \beta$ because, $\pi_2 \circ (\hat{u}(t), \pm\hat{v}(t)) = \pm\hat{v}(t) = p \circ \beta(t)$. We show, by contradiction, that it is indeed a horizontal lift. If it is not a horizontal lift, let $(\hat{u}'(t), \pm\hat{v}(t))$ be the unique horizontal lift with the initial value $\hat{u}'(0) = \hat{u}(0)$. It must have a shorter length than $(\hat{u}(t), \pm\hat{v}(t))$. It follows now that $(\hat{v}(t), \pm\hat{u}'(t))$ is a lift of β with a length shorter than $\tilde{\beta}(t) = (\hat{v}(t), \pm\hat{u}(t))$, and they have the same initial point i.e., $(\hat{v}(0), \pm\hat{u}'(0)) = (\hat{v}(0), \pm\hat{u}(0))$. This contradicts with the hypothesis that $\tilde{\beta}$ is a horizontal lift.

Thus, $\widetilde{p \circ \beta} = (\hat{u}(t), \pm\hat{v}(t))$ is a horizontal lift of $p \circ \beta$. Let us now consider lifts of $\tilde{\beta}$ and $\widetilde{p \circ \beta}$ in $L(2,1)$ i.e., $(\hat{v}(t), \hat{u}(t))$ and $(\hat{u}(t), \hat{v}(t))$ respectively. It is straightforward to see that the vertical displacements are identical and is given by the unique $SO(3)$ operator V which satisfies $V\hat{v}(0) = \hat{v}(1)$ and $V\hat{u}(0) = \hat{u}(1)$. ∎

We now return to prove theorem 2. Although the pieces β_j in S^2 cannot be attached continuously, their projections in $\mathbb{RP}^2$ can be attached continuously:

$$p \circ \beta_j(a_j) = \pm\frac{\dot{\gamma}(a_j)}{|\dot{\gamma}(a_j)|} = p \circ \beta_{j+1}(a_j) \tag{31}$$

This follows from Eq. 21. Indeed, the path obtained by attaching the segments $p \circ \beta_j$ in $\mathbb{RP}^2$ is α, the projection of γ defined in Eq. 5. From lemma 3, it follows that the vertical displacements of β_j and $p \circ \beta_j$ are equal. Thus, the vertical displacement of α is given by

$$V = R_{n+1} R_n \cdots R_1 \tag{32}$$

Where, R_j is the vertical displacement of β_j. This is equal to the geometric phase of γ, defined in Eq. 28.

C. Generalized Solid Angle

The notion of generalized solid angle was introduced through definition 3 in Section II. In the following, we show that this generalized solid angle reduces to the standard solid angle for non-singular loops. Furthermore, we discuss the reasons why this definition is a meaningful generalization of solid angles for singular loops. In particular, we discuss the case when the projected path α is open in $\mathbb{RP}^2$.

When γ is non-singular, its projection α is necessarily closed. We consider the following three cases separately — (i) γ is non singular, (ii) γ is singular and α is closed and (iii) γ is singular and α is an open path.

For a non-singular loop, by definition $|\gamma(t)| \neq 0$ throughout. Therefore it comprises of only one segment, i.e., $a_0 = 0$ and $a_1 = 1$. The corresponding projected paths in S^2, $\beta = \frac{\gamma}{|\gamma|}$ is closed. From lemma 3 and the definition of the geometric phase given by Eq. 28, it follows that the geometric phase (R) of γ is a rotation about $\beta(0)$ (or equivalently, about $\alpha(0)$) by an angle equal to the solid angle of γ. This angle is obtained by the expression $\cos^{-1}(\hat{k} \cdot R\hat{k})$ for some unit vector $\hat{k}$ normal to $\alpha(0)$. Thus, the generalized solid angle is consistent with the standard solid angle for non-singular loops.

For a singular loop, the standard solid angle is not well-defined. However, if the projection α is closed, i.e., $\alpha(0) = \alpha(1)$, the geometric phase (i.e., the vertical displacement of α) is still a rotation about $\alpha(0)$ — it maps the fiber above $\alpha(0)$ in $L(4,1)$ to itself. Therefore, the angle of rotation about $\alpha(0)$ is well-defined and is the natural extension of solid angles to this case.

Finally, we consider the case where α is open. Figure 3(b) shows one such example of a loop γ, whose projection is open in $\mathbb{RP}^2$. That is, $\gamma(0) = \gamma(1) = 0$ but $\pm\dot{\gamma}(0) = \alpha(0) \neq \alpha(1) = \pm\dot{\gamma}(1)$. Solid angles are well-defined for open paths in S^2 by closing them using a geodesic [22], [23]. We adopt a similar technique to define solid angles for open paths in $\mathbb{RP}^2$. The geometric phase (R) maps the fiber above $\alpha(0)$ to the fiber above $\alpha(1)$ in $L(4,1)$. Indeed, it can be written uniquely as a product of two rotations, one that takes $\alpha(0)$ to $\alpha(1)$ and another that rotates about $\alpha(1)$:

$$R = R_{\alpha(1)}(\Omega_2) R_{\hat{k}}(\Omega_1) \tag{33}$$

where $\hat{k}$ is a vector normal to $\alpha(0)$ and $\alpha(1)$ and Ω_1 is the angle between $\alpha(0)$ and $\alpha(1)$. The natural definition of solid angle for such a path is Ω_2, which is given by $\cos^{-1}(\hat{k} \cdot R\hat{k})$.

IV. CONCLUSIONS

We have shown that the geometrical properties of a loop traversed by the spin vector inside the Bloch ball can be extracted from the spin fluctuation tensor of a spin-1 quantum state. This property crucially depends on the Fubini-Study metric on $\mathbb{CP}^2$, and it reflects the deep synchrony between the geometry of real space and the geometry of the abstract space of quantum states. We have defined a geometric phase corresponding to each loop in a Bloch ball in the form of an $SO(3)$ operator to capture this effect. Furthermore, we have interpreted this geometric phase in terms of a generalized solid angle, defined for loops inside a Bloch ball.

Although we have considered a spin-1 system, our analysis can be generalized to any spin system. A spin-S system has independent moment tensors up to order $2S$. A natural extension of our work is to explore the geometric information carried by these higher order tensors.

One of the recent applications of geometric phase has been in characterizing topological phases of matter. Berry phase along a loop in the parameter space of a Hamiltonian is given by the integral of the Berry curvature evaluated over the region enclosed by the loop [8]. The total integral of the Berry curvature over the entire parameter space (usually the momentum space in condensed matter systems) is a topological invariant of the parameter space known as the *Chern number*. A topological phase transition is characterized by a "sudden change" of the the Chern number. Recent explorations [24], [25] have shown that mixed state generalizations of Berry's phase [4], [26] can also be used to characterize topological phase transitions. The geometric phase introduced in this paper could also be used to characterize topological states of 1-dimensional quantum systems.

This geometric phase can be measured using trapped atoms, ions or solid state systems through a controlled transport of the spin vector followed by a measurement of the fluctuations.

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[19] This picture has a generalization. $\mathbb{CP}^n$ is a *symmetrized product* of n $\mathbb{CP}^1$'s. It is the space of all symmetrized set of n points on a unit sphere. That is, $\mathbb{CP}^n = \mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1 / \sim$ where $(r_1, \cdots r_i, \cdots r_j, \cdots r_n) \sim (r_1, \cdots r_j, \cdots r_i, \cdots r_n)$ for $r_i \in \mathbb{CP}^1$. This is also a consequence of Majorana construction.

[20] We have used $\lim_{t \to a_k^\pm} \frac{\gamma_j(t)}{|\gamma_j(t)|} = \lim_{t \to a_k^\pm} \frac{\gamma_j(t) - \gamma_j(a_k)}{|\gamma_j(t) - \gamma_j(a_k)|} = \pm \frac{\dot{\gamma}_j(a_k)}{|\dot{\gamma}_j(a_k)|}$.

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